Lecture 8: Continuity and product spaces

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ x \mapsto x^2 + 1 \]

\[ g : \mathbb{R} \to \mathbb{R} \]
\[ x \mapsto 4x \]

Domain \((f) = \text{Domain} \,(g) = \mathbb{R} \)

Direct sum:
\[ f \oplus g : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \]
\[ x \mapsto (x^2 + 1, 4x) \in \mathbb{R}^2 \]

Generally, given \( f : X \to Y \), \( g : X \to Z \)

Here \((X, d_X), (Y, d_Y), (Z, d_Z)\) m-

Direct sum:
\[ f \oplus g : X \to Y \oplus Z \]
\[ x \mapsto (f(x), g(x)) \]

Direct sum operation preserves continuity.

Lemma: Let \( f : X \to \mathbb{R} \), \( g : X \to \mathbb{R} \) be functions and let \( f \oplus g : X \to \mathbb{R}^2 \) be their direct sum.
$(\mathbb{R}^2, \text{Euclidean metric})$

Then:

(a) If $x_0 \in X$ then $f$ and $g$ are cont. at $x_0$ iff $f \circ g$ is cont. at $x_0$.

(b) $f$ and $g$ are continuous iff $f \oplus g$ is continuous.

Proof:

(a) Let $(x^{(n)})_{n \geq 1}$ be any convergent sequence in $X$. Assume $x_0 = \lim_{n \to \infty} x_n \in X$.

Then $f$ and $g$ are cont. at $x_0$.

\[ \lim_{n \to \infty} f(x^{(n)}) = f(x_0), \quad \lim_{n \to \infty} g(x^{(n)}) = g(x_0) \]

in $(\mathbb{R}, d)$.

(b) $\forall n, \lim_{n \to \infty} (f(x^{(n)}) \oplus g(x^{(n)})) = (f(x_0) \oplus g(x_0))$ in $(\mathbb{R}^2, d_{\mathbb{R}^2})$ (Equivalence of $l^p$-metrics on $\mathbb{R}^n$).

(c) $\forall n, \lim_{n \to \infty} (f \oplus g)(x^{(n)}) = (f \oplus g)(x_0)$.

(f) $f \oplus g$ is cont. at $x_0$.

(b) obvious from (a).
Recall that:

- Addition function: \((x,y) \mapsto x+y\)
- Subtraction: \(s\)
- Maximum: \(\max(x,y)\)
- Minimum: \(\min(x,y)\)
- Division: \(R \times R \setminus \{0\} \to R\) \((x,y) \mapsto x/y\)
- Scalar mult.: \(x \mapsto cx\)

Corollary: \((X,d)\) m-s. Let \(f: X \to R\) be functions. Let \(c \in \mathbb{R}\). \(f: X \to R\).

(a) If \(x_0 \in X\) and \(f, g\) are cont. at \(x_0\) then \(f + g: X \to R\), \(fg: X \to R\), \(\max(f, g)\), \(\min(f, g)\), \(cf: X \to R\) are cont. at \(x_0\).

If \(g(x) \neq 0\) for all \(x \in X\) then \(f/g\) is also cont. at \(x_0\).

(b) Same statement for cont. everywhere.

Proof: (a) \(f + g = \circ \circ (f + g)\) (composition of 2 cont. fun. set)

\[ f - g = s \circ (f + g) \ldots \]

Also \(\tilde{f}/g = d \circ (f + g)\), where \(\tilde{g}: X \to R \setminus \{0\}\).
Remark: 1. \( f: X \rightarrow Y, \ g: X \rightarrow Z \)

\[ f \circ g: X \rightarrow Y \times Z \]

Here we do use the "product" \( l^2 \)-metric on \( Y \times Z \), i.e.,

\[ d_{Y \times Z}(y_1, z_1, y_2, z_2) = \sqrt{d_Y(y_1, y_2)^2 + d_Z(z_1, z_2)^2} \]

Then \( f, g \) are cont. on \( X \)

(=) \( f \circ g \) is cont. on \( X \).

2. \( f: X_1 \rightarrow Y, \ g: X_2 \rightarrow Y \times Z \)

Define \( f \oplus g: X_1 \times X_2 \rightarrow Y \times Z \)

\[ (x_1, x_2) \mapsto (f(x_1), g(x_2)) \]

\( f \oplus g \) is cont. on \( X_1 \times X_2 \) (=) both \( f \) and \( g \) are cont.

Special Case (Several Variables): \( f: R \rightarrow R^n \)

Then \( \oplus f = f \oplus f \oplus \cdots f \) \( R \rightarrow R^n \) i.e.,

\[ f(x) = (f_1(x), \ldots, f_n(x)) \]

The projection \( \pi_i: R^n \rightarrow R \)

\[ \pi_i: x \mapsto x_i \]

Then \( f_i = \pi_i \circ f \), \( i \leq n \)
Note that $\pi_i$ is continous for all $i$, so $f$ is continous if $f_i$ is continous.

Also, if $g_j : \mathbb{R} \to \mathbb{R}$ is continous, then $g_j(x) = e^{x_j}$ is continous for all $1 \leq j \leq m$.

Let $g : \mathbb{R}^m \to \mathbb{R}$ given by $(x_1, \ldots, x_m) \mapsto g_1(x_1)g_2(x_2) \cdots g_m(x_m)$.

Thus, all polynomials with real coefficients

$$p(x) = \sum_{\alpha \in (\mathbb{N}^+)^m} c_\alpha x_1^{\alpha_1} \cdots x_m^{\alpha_m}$$

are continous on $\mathbb{R}^m$.

If $f : X \to \mathbb{R}$ is continous, then $|f| : X \to \mathbb{R}$ is also continous because $|f| = f^+ + f^-$

where $f^+ = \max(f, 0)$ and $f^- = \min(f, 0)$.

Use notation

$$C(X, \mathbb{R}) = \{ f : X \to \mathbb{R} \mid f \text{ is continous} \}$$

By Corollary 1, $C(X, \mathbb{R})$ is an algebra.