Exercise 1.1.3

Definition 1.1. Metric Spaces A metric space \((X, d)\) is a space \(X\) of objects (called points), together with a distance function or metric \(d : X \times X \to [0, \infty)\), which associates to each pair \(x, y\) of points in \(X\) a non-negative real number \(d(x, y) \geq 0\). Furthermore, the metric must satisfy the following four axioms:

a \quad \forall x \in X, d(x, x) = 0

b \quad \forall x, y \in X, x \neq y \implies d(x, y) > 0

c \quad \forall x, y \in X, d(x, y) = d(y, x)

d \quad \forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)

Let \(X\) be a set, and let \(d : X \times X \to [0, \infty)\) be a function.

A. Give an example of a pair \((X, d)\) which obeys axioms \((bcd)\) of Definition 1.1, but not \((a)\). (Hint: modify the discrete metric.)

Consider \((\mathbb{R}, \hat{d})\) where \(\hat{d} : \mathbb{R} \times \mathbb{R} \to [0, \infty)\), is defined by,

\[
\hat{d}(x, y) = 1 \quad \forall x, y \in \mathbb{R}
\]

Then by construction, \((a)\) is not satisfied since \(\hat{d}(x, x) = 1\). Next we show \((bcd)\) is obeyed. Firstly, \(\forall x, y \in \mathbb{R}, \hat{d}(x, y) > 0\), thus \((b)\) is satisfied. Secondly, \(\hat{d}\) is symmetric since the metric takes on only one value, hence, \(\forall x, y \in \mathbb{R}, \hat{d}(x, y) = \hat{d}(y, x)\). Lastly, we show the triangle equality. Suppose \(x, y, z \in \mathbb{R}\). Then we have,

\[
\hat{d}(x, y) + \hat{d}(y, z) = 1 + 1 = 2 \\
\geq \hat{d}(x, z) = 1
\]

B. Give an example of a pair \((X, d)\) which obeys axioms \((acd)\) of Definition 1.1, but not \((b)\).

Consider \((\mathbb{R}, \hat{d})\) where \(\hat{d} : \mathbb{R} \times \mathbb{R} \to [0, \infty)\), is defined by,

\[
\hat{d}(x, y) = 0 \quad \forall x, y \in \mathbb{R}
\]

Then by construction, \((b)\) is not satisfied since \(\forall x, y \in \mathbb{R}, \hat{d}(x, y) \neq 0\). Next we show \((acd)\) is obeyed. Firstly, \(\forall x, y \in \mathbb{R}, \hat{d}(x, y) = 0\), thus \((a)\) is satisfied. Secondly, the metric is symmetric since the metric takes on only one value. Lastly, we show the triangle equality. Suppose \(x, y, z \in \mathbb{R}\). Then we have,

\[
\hat{d}(x, y) + \hat{d}(y, z) = 0 + 0 = 0 \\
\geq \hat{d}(x, z) = 0
\]

The last step identifies that \(\hat{d}(x, z) = 0\) and trivially \(0 \geq 0\).
C Give an example of a pair \((X, d)\) which obeys axioms (abd) of Definition 1.1, but not (c).

Consider \((\mathbb{R}, \hat{d})\) where \(\hat{d}: \mathbb{R} \times \mathbb{R} \to \{0, \frac{1}{2}, 1\}\), where \(\hat{d}\) is defined as follows, \(\forall x, y \in \mathbb{R}\),

\[
\hat{d}(x, y) = \begin{cases} 
1 & x < y \\
\frac{1}{2} & x > y \\
0 & x = y 
\end{cases}
\]

First we show (c) is not satisfied. Suppose \(x, y \in \mathbb{R}\) and w.l.o.g, suppose \(x > y\). Then \(\hat{d}(x, y) = \frac{1}{2} \neq 1 = \hat{d}(y, x)\). Then, (c) is not satisfied. Next we show (abd) is obeyed. Firstly, \(\forall x \in \mathbb{R}, \hat{d}(x, x) = 0\), thus (a) is satisfied. Secondly, \(\hat{d}\) takes on values strictly non-negative, hence the positivity axiom (b) is satisfied. Lastly we show the triangle inequality (d).

Suppose \(x, y, z \in \mathbb{R}\) and are distinct. Then we have,

\[
\hat{d}(x, y) + \hat{d}(y, z) \geq \frac{1}{2} + \frac{1}{2} = 1 \\
\geq \hat{d}(x, z)
\]

If \(x = z\), or \(x = z = y\) then trivially the inequality is satisfied since \(\hat{d}(x, z) = \hat{d}(x, x) = 0\) and any sum of non-negative values is greater than or equal to 0. Moreover, if \(y = x\) or \(z = y\) the inequality is also satisfied since

\[
\hat{d}(x, y) + \hat{d}(y, z) = \hat{d}(x, x) + \hat{d}(x, z) \\
= \hat{d}(x, z) \\
\geq \hat{d}(x, z)
\]

and

\[
\hat{d}(x, y) + \hat{d}(y, z) = \hat{d}(x, z) + \hat{d}(x, z) \\
= 2\hat{d}(x, z) \\
\geq \hat{d}(x, z)
\]

respectively.

D Give an example of a pair \((X, d)\) which obeys axioms (abc) of Definition 1.1, but not (d). (Hint: try examples where \(X\) is a finite set.)

Consider \((X, \hat{d})\) where \(X = \{1, 2, 3\}\), and \(\hat{d}: X \times X \to \{0, 1, 3\}\), where \(\hat{d}\) is defined by the table which follows, \(\forall x, y \in X\),

\[
\begin{array}{ccc}
\text{x} & \text{y} & 1 & 2 & 3 \\
1 & 0 & 3 & 1 \\
2 & 3 & 0 & 1 \\
3 & 1 & 1 & 0 \\
\end{array}
\]

First we show (d) is not satisfied. Consider \(d(1, 2) = 3 \neq 2 = d(1, 3) + d(3, 2)\) thus this is an example of a violation of the triangle inequality thus, (c) is not satisfied. Next we show (abc) is obeyed. Firstly, \(\forall x \in X, \hat{d}(x, x) = 0\), thus (a) is satisfied. Secondly, the \(\hat{d}\) takes on values strictly non-negative, hence the positivity axiom (b) is satisfied. Lastly we show the symmetry (c). The above table is symmetric about the \(y = x\) diagonal of zeros, thus the metric is defined symmetrically and (c) is satisfied by construction.

**2 Exercise 1.1.5**

Let \(n \geq 1\), and let \(a_1, a_2, ..., a_n\) and \(b_1, b_2, ..., b_n\) be real numbers. Verify the identity:

\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2 = \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{j=1}^{n} b_j^2 \right)
\]
and conclude the Cauchy-Schwarz inequality

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{j=1}^{n} b_j^2 \right)^{1/2}$$

Then use the Cauchy-Schwarz inequality to prove the triangle inequality

$$\left( \sum_{i=1}^{n} (a_i + b_i)^2 \right)^{1/2} \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} + \left( \sum_{j=1}^{n} b_j^2 \right)^{1/2}$$

### 2.1

We begin by verifying the identity, writing (I) + (II) = (III) for the given terms of the equation:

**Proof.**  
1. By algebra, the first term (I) is equal to a sum of two series, call them $\sigma_1$ and $\sigma_2$.

$$\left( \sum_{i=1}^{n} a_i b_i \right)^2 = \sum_{i=1}^{n} a_i^2 b_i^2 + 2 \sum_{i=1}^{n} a_i b_i a_j b_j$$

$$= \sigma_1 + \sigma_2$$

2. By algebra, the second term (II) can be written as a sum of $-\sigma_2$ and another term $\sigma_3$.

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2 = -2 \sum_{i=1}^{n} \sum_{j \neq i}^{n} a_i b_i a_j b_j + \sum_{i=1}^{n} \sum_{j \neq i}^{n} a_i^2 b_j^2$$

$$= -\sigma_2 + \sigma_3$$

3. The product of two series (III) is equal to the sum of $\sigma_1$ and $\sigma_3$.

$$\left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{j=1}^{n} b_j^2 \right) = \sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{i=1}^{n} \sum_{j \neq i}^{n} a_i^2 b_j^2$$

$$= \sigma_1 + \sigma_3$$

4. Moreover, $\sigma_1 + \sigma_3 = (\sigma_1 + \sigma_2) + (-\sigma_2 + \sigma_3)$.

By the above, we have shown that (I) + (II) = (III).

\[ \square \]

### 2.2

(From 2.1, conclude the Cauchy-Schwarz inequality.)

**Proof.**  
1. First, it is obvious that

$$\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2.$$  

2. Using the identity from 2.1, we obtain:

$$\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{j=1}^{n} b_j^2 \right).$$

3. Since all terms are positive we can take the square root, preserving order:

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{j=1}^{n} b_j^2 \right)^{1/2}.$$

\[ \square \]
2.3

Prove:

\[
\left( \sum_{i=1}^{n} (a_i + b_i)^2 \right)^{1/2} \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} + \left( \sum_{j=1}^{n} b_j^2 \right)^{1/2}
\]

Proof. We will refer to the left-hand side of the above inequality as (I) and the right-hand side as (II).

1. Note that

\[
(II)^2 = \sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{n} b_j^2 + 2 \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2}
\]

2. From 2.2, we can conclude that

\[
\sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{n} b_j^2 + 2 \left| \sum_{i=1}^{n} a_i b_i \right| \leq (II)^2.
\]

3. Note that

\[
(I)^2 = \sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{n} b_j^2 + 2 \sum_{i=1}^{n} a_i b_i.
\]

Moreover,

\[
\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \left| \sum_{i=1}^{n} a_i b_i \right|
\]

4. By statements 2 and 3, we know

\[
(I)^2 \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \left| \sum_{i=1}^{n} a_i b_i \right| \leq (II)^2
\]

5. Thus (I) \(\leq\) (II).

\(\square\)

3 Exercise 1.1.6

Show that \((\mathbb{R}^n, d_{l_2})\) is indeed a metric space.

Proof. Let \(x, y, z \in \mathbb{R}^n\)

(a)

\[d_{l_2}(x, x) = \left( \sum_{i=1}^{n} (x_i - x_i)^2 \right)^{1/2} = \left( \sum_{i=1}^{n} (0)^2 \right)^{1/2} = 0\]

(b) suppose \(x \neq y\), thus \(\exists k \in \mathbb{N}\) st \(1 \leq k \leq n\) and \(x_k \neq y_k\)

\[d_{l_2}(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \geq \left| x_k - y_k \right| > 0\]

(c) as

\[(x_i - y_i)^2 = (y_i - x_i)^2,\]

\[d_{l_2}(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} = \left( \sum_{i=1}^{n} (y_i - x_i)^2 \right)^{1/2} = d_{l_2}(y, x)\]

(d)

\[\left( \sum_{i=1}^{n} (x_i - z_i)^2 \right)^{1/2} \leq \left( \sum_{i=1}^{n} (x_i - y_i + y_i - z_i)^2 \right)^{1/2} \leq \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} + \left( \sum_{j=1}^{n} (y_j - z_j)^2 \right)^{1/2}\]

By the property proved in 1.1.5

\(\square\)
4 Exercise 1.1.11
Show that \((\mathbb{R}^n, d_{\text{disc}})\) is indeed a metric space.

Proof. (ab) holds by construction.
(c) holds as equality is a symmetric relation
(d) holds by cases. Suppose \(x=\). Thus \(d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) = 0 = d_{\text{disc}}(x, z)\)
Suppose \(x \neq \). Thus \(y \neq \) or both (or else \(x=\) which is false by assumption), so \(d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) = 1 \vee 2 \geq 1 = d_{\text{disc}}(x, z)\) \(\square\)

5 Exercise 1.1.12

Proposition 1. Let \(\mathbb{R}^n\) be a Euclidean space, and let \((x^{(k)})_{k=m}^\infty\) be a sequence of points in \(\mathbb{R}^n\). We write \(x^{(k)} = (x_1^{(k)}, x_2^{(k)}, ..., x_n^{(k)})\). Let \(x = (x_1, ..., x_n)\) be a point in \(\mathbb{R}^n\). Then the following four statements are equivalent:

a \((x^{(k)})_{k=m}^\infty\) converges to \(x\) with respect to the Euclidean metric \(d_2\)

b \((x^{(k)})_{k=m}^\infty\) converges to \(x\) with respect to the taxi-cab metric \(d_1\)

c \((x^{(k)})_{k=m}^\infty\) converges to \(x\) with respect to the sup norm metric \(d_{\sup}\)

d For every \(1 \leq j \leq n\), the sequence \((x_j^{(k)})_{k=m}^\infty\) converges to \(x_j\).

5.1
Proof. We will show \((a) \implies (d) \implies (b) \implies (c) \implies (a)\).

1. \((a \implies d)\): Suppose that \(\lim_{k \to \infty} \left[ \sum_{i=1}^{n} \left( x_i^{(k)} - x_i \right)^2 \right]^{1/2} = 0\). Then for each \(1 \leq j \leq n\), we have

\[
0 = \lim_{k \to \infty} \left[ \sum_{i=1}^{n} \left( x_i^{(k)} - x_i \right)^2 \right]^{1/2} \geq \lim_{k \to \infty} \left[ \left( x_j^{(k)} - x_j \right)^2 \right]^{1/2} = \lim_{k \to \infty} |x_j^{(k)} - x_j|.
\]

So \(\lim_{k \to \infty} |x_j^{(k)} - x_j| = 0\) and (d) holds.

2. \((d \implies b)\): Suppose that for every \(1 \leq j \leq n\), the sequence \((x_j^{(k)})_{k=m}^\infty\) converges to \(x_j\). Then \(\sum_{i=1}^{n} |x_i^{(k)} - x_i|\) is a sum of finitely many convergent sequences, and thus converges as \(k \to \infty\). In particular,

\[
\lim_{k \to \infty} \sum_{i=1}^{n} |x_i^{(k)} - x_i| = \sum_{i=1}^{n} \lim_{k \to \infty} |x_i^{(k)} - x_i| = 0.
\]

So (b) holds.

3. \((b \implies c)\): Suppose that \(\lim_{k \to \infty} \sum_{i=1}^{n} |x_i^{(k)} - x_i| = 0\). Then we have

\[
0 = \lim_{k \to \infty} \sum_{i=1}^{n} |x_i^{(k)} - x_i| \geq \lim_{k \to \infty} \sup_{1 \leq i \leq n} |x_i^{(k)} - x_i|
\]

So \(\lim_{k \to \infty} \sup_{1 \leq i \leq n} |x_i^{(k)} - x_i| = 0\), and (c) holds.

4. \((c \implies a)\): Suppose that \(\lim_{k \to \infty} \sup_{1 \leq i \leq n} |x_i^{(k)} - x_i| = 0\). Then we have

\[
\lim_{k \to \infty} \left[ \sum_{i=1}^{n} \left( x_i^{(k)} - x_i \right)^2 \right]^{1/2} \leq \lim_{k \to \infty} \left[ \sum_{i=1}^{n} \left( \sup_{1 \leq i \leq n} |x_i^{(k)} - x_i| \right)^2 \right]^{1/2} = \lim_{k \to \infty} \sum_{i=1}^{n} |x_i^{(k)} - x_i| = 0.
\]

So \(\lim_{k \to \infty} \left[ \sum_{i=1}^{n} \left( x_i^{(k)} - x_i \right)^2 \right]^{1/2} = 0\) and (a) holds. \(\square\)
Exercise 1.1.16

Let \((x_n)_{n=1}^\infty\) and \((y_n)_{n=1}^\infty\) be two sequences in a metric space \((X,d)\). Suppose that \((x_n)_{n=1}^\infty\) converges to a point \(x \in X\) and \((y_n)_{n=1}^\infty\) converges to a point \(y \in X\). Show that \(\lim_{n \to \infty} d(x_n, y_n) = d(x, y)\).

Proof. Suppose the above hypothesis. Then, \(\forall \epsilon > 0, \exists N > 0 \in \mathbb{N} \text{ s.t. } \forall n \geq N \text{ and } \exists M > 0 \in \mathbb{N} \text{ s.t. } \forall m \geq M, \) the following two hold

\[
d(x_n, x) \leq \epsilon
\]

\[
d(y_m, y) \leq \epsilon
\]

Then \(\forall n \geq \max\{N, M\}\), by two applications of the triangle inequality, we have

\[
d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y_n, y)
\]

\[
= 2\epsilon + d(x, y)
\]

\[
d(x_n, y_n) - d(x, y) \leq 2\epsilon
\]

also, analogously, by again applying the triangle inequality twice,

\[
d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)
\]

\[
= 2\epsilon + d(x_n, y_n)
\]

thus,

\[
d(x, y) - d(x_n, y_n) \leq 2\epsilon
\]

hence,

\[
|d(x, y) - d(x_n, y_n)| \leq 2\epsilon
\]

so \(\lim_{n \to \infty} d(x_n, y_n) = d(x, y)\).