Instructions to Candidates

1. This examination paper contains FIVE questions and comprises NINE (9) printed pages.

2. Answer ALL the questions in the space provided. The points for each question are indicated at the beginning of each question.

3. Solutions must include all work and be shown in a clear organized manner.

4. Answers should always be in EXACT and SIMPLIFIED form unless you are specifically asked to estimate.

5. All cell phones and all electronic devices that can receive or transmit wireless signal must be turned off during the entire exam.

6. Approved calculators are permitted.

7. Use algebra for calculations, NOT built-in features of your calculator.
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Question 1. [30 points]

(I) Use row reduction method to find the inverse of $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ if it exists. (10 points)

(II) Let $H$ be the following domain in the $xy$-plane:

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq y \geq 1 \right\}.$$ 

Is $H$ a subspace of $\mathbb{R}^2$? Explain. (5 points)

(III) Let $A$, $B$ be two $n \times n$ matrices such that $\det(AB^2) = 1$. Show that $A$ and $B$ are invertible and moreover,

$$\det(B) = \frac{1}{\sqrt{\det(A)}}$$

(10 points)
Let $A$ be a $5 \times 6$ matrix. Recall that $A^T$ is the transpose of $A$. Apply the Rank Theorem to $A$ and $A^T$ to show that

$$\dim \operatorname{Nul}(A) = \dim \operatorname{Nul}(A^T) + 1$$

(5 points)

**Question 2.**

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}.$$ 

Is $T$ invertible? Explain why or why not. If $T$ is invertible, then find a formula for $T^{-1}$ in terms of $x_1$ and $x_2$. 

[10 points]
Question 3. [10 points]
Consider the following subspace of \( \mathbb{R}^3 \).

\[
H = \left\{ \begin{bmatrix} a + b \\ 2b \\ 2a + b \end{bmatrix} : a, b \text{ are real numbers} \right\}
\]

(i) Find a basis for \( H \).

(ii) Find the dimension of \( H \).
Question 4. [20 points]

Let $A = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix}$.

(i) Find a basis for the null space of $A$. Justify your answer.

(ii) Find a basis for the column space of $A$. Justify your answer.
(iii) Find a basis for the row space of $A$. Justify your answer.

(iv) What is the rank of $A$? Justify your answer.
Question 5. [30 points]

(A) Find whether the following statements are True or False. Circle your answer. (3 points each)

(i) Let $A, B, C$ be three $9 \times 9$ matrices. If $ABC$ is invertible, then $C$ is also invertible.
   
   \begin{align*}
   (a) & \text{ True} \\
   (b) & \text{ False}
   \end{align*}

(ii) For any $n \times n$ invertible matrix $P$, $(PAP^{-1})^2 = PA^2P^{-1}$ for any $n \times n$ matrix $A$.
   
   \begin{align*}
   (a) & \text{ True} \\
   (b) & \text{ False}
   \end{align*}

(iii) $(A - B)^2 = A^2 - 2AB + B^2$ for any $n \times n$ matrices $A, B$.
   
   \begin{align*}
   (a) & \text{ True} \\
   (b) & \text{ False}
   \end{align*}

(iv) Let $V$ be a finite-dimensional vector space. If $\{v_1, v_2, v_3\}$ is a linear dependent set in $V$, then $\dim V < 3$.
   
   \begin{align*}
   (a) & \text{ True} \\
   (b) & \text{ False}
   \end{align*}

(v) If $A, B$ are invertible $5 \times 5$ matrices then $\det(A + B) = \det(A) + \det(B)$.
   
   \begin{align*}
   (a) & \text{ True} \\
   (b) & \text{ False}
   \end{align*}

(vi) If $A, B$ are $5 \times 5$ matrices such that $\det(BA) = 1$ then $B$ must be the inverse of $A$.
   
   \begin{align*}
   (a) & \text{ True} \\
   (b) & \text{ False}
   \end{align*}

(vii) If the matrices $A$ and $B$ have the same reduced row echelon form then $\text{Col}(A) = \text{Col}(B)$.
   
   \begin{align*}
   (a) & \text{ True} \\
   (b) & \text{ False}
   \end{align*}
(B) For each of the following questions, circle your answer.

(i) If the null space of a $7 \times 5$ matrix $A$ is 3-dimensional, what is the dimension of the row space of $A$? (4 points)

(a) 0   (b) 1
(c) 2   (d) 3
(e) 4   (f) 5
(g) 6   (h) 7

(ii) Let $B$ be a $3 \times 3$ matrix. What are the all possible dimensions for $\text{Nul}(B)$? (5 points)

(a) 0   (b) 0, 1
(c) 0, 1, 2   (d) 0, 1, 2, 3
(e) 1   (f) 1, 2
(g) 1, 2, 3   (h) 2
(i) 2, 3   (j) 3